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# Stationarity–conservation laws for fractional differential equations with variable coefficients

**Małgorzata Klimek**

Institute of Mathematics and Computer Science, Technical University of Częstochowa, ul Dąbrowskiego 73, 42-200, Częstochowa, Poland

E-mail: klimek@matinf.pcz.czest.pl

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**Abstract**

In this paper, we study linear fractional differential equations with variable coefficients. It is shown that, by assuming some conditions for the coefficients, the stationarity–conservation laws can be derived. The area where these are valid is restricted by the asymptotic properties of solutions of the respective equation. Applications of the proposed procedure include the fractional Fokker–Planck equation in (1 + 1)- and ($d$ + 1)-dimensional space and the fractional Klein–Kramers equation.

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## 1. Introduction

The aim of this paper is to derive conserved currents for fractional differential equations. For differential equations built within the framework of classical differential calculus, the conservation laws are connected via the Noether theorem [1, 2] with the variational symmetry of the action. In the modern theory of differential equations, methods of differential geometry have also been developed and applied in the classification and derivation of conservation laws (see [2–5], and references therein). For linear differential equations with constant coefficients, the conserved currents can also be obtained using the simple procedure proposed by Takahashi and Umezawa [6, 7] by which an explicit expression for a conserved current is calculated without recourse to the Lagrange formalism.

They show that the solution of the operator equation

$$\sum_{\mu}(\overleftarrow{\partial}^{\mu} + \partial^{\mu})\Gamma_{\mu} = \Lambda(\partial) - \Lambda(-\overleftarrow{\partial}) \tag{1}$$

together with solutions $\Phi$ and $\Phi'$ of initial and conjugated equations

$$\Lambda(\partial)\Phi = 0 \qquad \Phi'\Lambda(-\overleftarrow{\partial}) = 0 \tag{2}$$

yield components of the current obeying the conservation law

$$J_\mu = \Phi' \Gamma_\mu \Phi \qquad \sum_\mu \partial^\mu J_\mu = 0. \tag{3}$$

In this paper, we extend this method to fractional differential equations with variable coefficients and we obtain stationarity–conservation laws. Fractional differential calculus has been applied recently in some areas of physics. Nevertheless, the geometrical and coordinate independent methods are not sufficiently developed for differential equations including derivatives of fractional order, so we have chosen to apply an extension of the technique which has proven to be useful also in the case of noncommutative and discrete models [8–11]. It has been shown in these papers that ideas from classical differential calculus on Minkowski space can also be applied in discrete and noncommutative differential calculus. The difference lies in the deformation of Leibniz's rule, where the additional transformation operator appears. Due to this fact the modification of the classical method was necessary and by this modified procedure we have derived conservation laws and an explicit form of conserved currents for discrete and noncommutative models with constant and variable coefficients.

The concept of derivatives and integrals of fractional order dates back to the time of the beginning of differential calculus and is connected with ideas and work by l'Hôpital, Leibniz, Euler and Laplace. Since then, the extensive study of properties of fractional differential and integral calculus has revealed similarities and differences between fractional and classical calculi. The wide scope of fundamental and useful results is covered in monographies [12–16] and references therein.

Section 2 contains a brief review of the fundamental properties of fractional operators together with a construction of Leibniz's rule in the convolution algebra of functions. We work with Riemann–Liouville-type derivatives and integrals; while Leibniz's rule is complicated for standard point-wise multiplication, it is simplified for convolution algebra and allows an extension of the Takahashi–Umezawa method to fractional equations.

In an earlier paper [17] we have investigated the construction of conservation laws for fractional linear differential equations with constant coefficients. We now proceed to fractional linear equations with variable coefficients subject to certain restrictions (24)–(26) which we discuss in section 3.

We should point out that fractional differential calculus has recently found application in various areas of physics [16, 18, 19]. The field where many interesting fractional equations have been derived and studied in various aspects is anomalous transport. Starting with early results for anomalous diffusion described by the fractional diffusion equation discussed by Nigmatullin [20], Mainardi [21], Wyss and Schneider [22, 23], the fractional derivatives also appear in the generalized equation of diffusion. The latter equation has been studied in the context of transport on fractals by O'Shaughnessy and Proccacia [24], Giona and Roman [25] as well as by Metzler and co-workers [26–28]. The properties of solutions discussed in these papers show that their asymptotic behaviour differs from the classical diffusion equation and it coincides with some experimental data.

The fractional diffusion equation arises also as the limiting dynamic equation for all continuous time random walks with decoupled temporal and spatial memories and with either temporal or spatial scale invariance [29, 30]. The diffusion equation then contains the Riemann–Liouville time fractional derivative or the Riesz spatial one, respectively. The model including both temporal and spatial fractional derivatives has been discussed recently by Barkai [31] where the validity domain of the fractional diffusion equation was addressed.

When transport phenomena are investigated in the presence of the external force field, we arrive at fractional equations with variable coefficients.

One of the first and most widely investigated is the Fokker–Planck equation in fractional formulation. This contains a fractional time derivative of Riemann–Liouville-type [32–38] when the force field depends only on spatial coordinates. Time-dependent external fields have been treated by Sokolov *et al* [39]; they assumed that the force field can be switched on and off and they derived the fractional Fokker–Planck equation. This is expected to be applicable to polymers and rough interfaces [40].

Fractional calculus is also useful in the context of Lévy flights which constitute a generalization of ordinary Brownian walks. The step size in such a model is drawn from Lévy distribution characterized by the step index. The built-in superdiffusive character of Lévy flights has been used to model a variety of physical processes. In the presence of a quenched random force field with arbitrary range and vector characteristic, the Fokker–Planck equation for Lévy flights contains a fractional gradient operator with respect to spatial coordinates [41–43].

The Brownian system subjected to a Lévy stable random force is described also in [44] where fractional extension of the Klein–Kramers equation with Riesz fractional derivative in momentum coordinates produces the Riemann–Liouville derivative with respect to the position in Fokker–Planck equation. The fractional Klein–Kramers equation in phase space has also been discussed in [36, 45].

This brief review of some of the results concerning application of fractional differential calculus shows that fractional derivatives are indeed present and useful in modern transport theory. We focus throughout the paper on equations with Riemann–Liouville-type derivatives. The general method of derivation of the stationarity–conservation laws is described in section 3. For a class of fractional differential equations, we also discuss the stationary and conserved charges resulting from the assumption that for a given model the stationarity–conservation law is valid for the whole space. Section 4 contains applications of the developed procedure to fractional versions of Fokker–Planck, Klein–Kramers, Cattaneo and generalized diffusion equations.

## 2. Fractional integrals and derivatives

### 2.1. Riemann–Liouville fractional integral and derivative for functions of one variable

Let us recall the definition of the Riemann–Liouville fractional integral [12–15] used widely in the literature dealing with fractional calculus.

**Definition 2.1.** *Let* $\mathrm{Re}\,\nu > 0$ *and let* $f$ *be piece-wise continuous on* $(0, +\infty)$ *and integrable on any finite subinterval of* $[0, +\infty)$. *Then for* $t > 0$

$$D_t^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s)\,\mathrm{d}s \tag{4}$$

*is the Riemann–Liouville fractional integral of* $f$ *of the order* $\nu$.

We notice that the above definition includes the operation of the Laplace convolution, namely it can be written as

$$D_t^{-\nu} f(t) = \Phi_{-\nu} * f(t) = f * \Phi_{-\nu}(t) \tag{5}$$

where we have denoted $\Phi_{-\nu}(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1}$.

The fractional integral satisfies the composition rule which is the generalization of the Dirichlet formula [12–15]

$$D_t^{-\nu} D_t^{-\mu} f(t) = D_t^{-(\nu+\mu)} f(t) = D_t^{-\mu} D_t^{-\nu} f(t) \tag{6}$$

for $\mathrm{Re}\,\mu, \mathrm{Re}\,\nu > 0$ and for any function $f$ piece-wise continuous on $[0, +\infty)$.

Let us now present the known forms of Leibniz's rule for fractional integral (4)

$$D_t^{-\nu}(f \cdot g) = \sum_{j=0}^{\infty} \binom{-\nu}{j} D_t^{-\nu-j} f \cdot g^{(j)} \tag{7}$$

where $f$ and $g$ are real analytic functions on $[0, +\infty)$. This rule was generalized by Osler [12, 46–48] who obtained the following forms of Leibniz's rule:

$$D_t^{-\nu}(f \cdot g) = \sum_{j=-\infty}^{+\infty} \frac{\Gamma(-\nu+1)}{\Gamma(-\nu-\gamma-j+1)\Gamma(\gamma+j+1)} D_t^{-\nu-\gamma-j} f \cdot D_t^{\gamma+j} g \tag{8}$$

$$D_t^{-\nu}(f \cdot g) = \int_{-\infty}^{+\infty} \frac{\Gamma(-\nu+1)}{\Gamma(-\nu-\gamma-\lambda+1)\Gamma(\gamma+\lambda+1)} D_t^{-\nu-\gamma-\lambda} f \cdot D_t^{\gamma+\lambda} g \, \mathrm{d}\lambda \tag{9}$$

where $\gamma$ is an arbitrary complex number.

We notice that when the algebra of functions is defined by standard point-wise multiplication, as in the above formulae, all versions of Leibniz's rule are very complicated.

Thus, we have proposed [17] to investigate the algebra of functions with multiplication defined via the Laplace convolution

$$f * g(t) := \int_0^t f(t-s) g(s) \, \mathrm{d}s. \tag{10}$$

As is well known, this multiplication is associative and commutative. The neutral element is the Dirac $\delta$-function. In this algebra the following rule is valid for $\mathrm{Re}\,\nu > 0$ and $\gamma$ is a complex number fulfilling $\mathrm{Re}(\nu - \gamma) \geqslant 0$ [17]:

$$D_t^{-\nu}(f * g) = \left(D_t^{-(\nu-\gamma)} f\right) * D_t^{-\gamma} g. \tag{11}$$

Thus, Leibniz's rule in algebra (10) is deformed (nevertheless much simpler than the above Leibniz's rules in the algebra of point-wise multiplication)

$$D_t^{-\nu}(f * g) = \beta \left(D_t^{-\nu} f\right) * g + (1-\beta) f * \left(D_t^{-\nu} g\right) \tag{12}$$

where $\beta \in [0, 1]$.

The fractional derivation is connected with the fractional integral (4). The operator known as the Riemann–Liouville fractional derivative [12–15] is defined as follows.

**Definition 2.2.** *Let $m \leqslant \mathrm{Re}\,\nu < m+1$, $t > 0$. The operator given by formula*

$$D_t^{\nu} := \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} D_t^{-(m-\nu+1)} f(t) \tag{13}$$

*for functions for which the improper integral on the right-hand side of equation (13) is convergent, is called the Riemann–Liouville fractional derivative of the order $\nu$.*

Let us notice that the functions from the domain of the $D_t^{\nu}$ operator form the subset in the set of functions from definition 2.1. It is well described in the literature [12–15].

Contrary to the fractional integrals, the derivative (13) cannot be expressed using only convolution. The formula includes the classical derivative and looks as follows,

$$D_t^{\nu} f(t) := \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} (f * \Phi_{\nu-m}(t)) \tag{14}$$

with the function $\Phi_{\nu-m} = \frac{t^{-\nu+m}}{\Gamma(m+1-\nu)}$.

We expect the fractional derivative to obey the composition rule analogous to that for the fractional integral. In fact [12, 13] the following formula is valid,

$$D_t^{\nu} D_t^{\mu} f = D_t^{\nu+\mu} f \tag{15}$$

provided:

- $\operatorname{Re}\nu > 0, \operatorname{Re}(\nu+\mu) > 0$;
- $f$ is of the form $f = D_t^{-(\nu+\mu)}\psi$ where $\psi \in L_1[0, b]$ which means $\int_0^b \mid \psi(t) \mid \mathrm{d}t < \infty$.

The above formula shows that fractional derivatives of different orders do not always commute as is the case with fractional integrals.

Leibniz's rule for fractional derivative has the following form for analytic functions [12, 13, 46–48]

$$D_t^{\nu} f \cdot g = \sum_{k=0}^{\infty} \binom{\nu}{k} f^{(k)} \cdot D_t^{\nu-k} g. \tag{16}$$

Similarly to the case of the fractional integral, for the fractional derivative Leibniz's rule can also be simplified in the algebra of functions (10). To this aim, the following Lemma is applied [17].

**Lemma 2.1.** *Let $m \leqslant \operatorname{Re}\nu < m+1$ and the function $g$ be piece-wise continuous in $(0, +\infty)$. If the fractional derivative of function $f$ does exist and the function itself fulfils the condition*

$$\lim_{t\to 0^+} f^{(k)} * \Phi_{\nu-m} = 0$$

*for $k = 0, 1, \ldots, m$ then the following rule holds:*

$$D_t^{\nu}(f * g) = \left(D_t^{\nu} f\right) * g. \tag{17}$$

The above set of right-sided limits determines the behaviour of the function $f$ in the neighbourhood of $t = 0$, namely $f(t) \sim t^{\beta}$ where $\beta$ is a complex number fulfilling the condition, $\operatorname{Re}\beta > -1 + \operatorname{Re}\nu$.

The symmetric version of formula (17) follows from the commutativity of the Laplace convolution.

**Corollary 2.2.** *We let $m \leqslant \operatorname{Re}\nu < m+1$ and functions $f$ and $g$ are piece-wise continuous in $(0, +\infty)$. If both functions $f, g$ obey the assumptions from lemma 2.1 then the following rule holds,*

$$D_t^{\nu}(f * g) = \beta\left(D_t^{\nu} f\right) * g + (1-\beta) f * \left(D_t^{\nu} g\right) \tag{18}$$

*for $\beta \in [0, 1]$.*

### *2.2. Riemann–Liouville partial fractional derivatives*

For functions of many variables, the partial fractional derivative of Riemann–Liouville-type is defined by the formula [15, 17]

$$D_k^{\alpha_k} f(\vec{x}) := \frac{1}{\Gamma(m_k+1-\alpha_k)} \left(\partial_{x_k}\right)^{m_k+1} \int_0^{x_k} (x_k - s)^{-\alpha_k+m_k} f(\vec{x} + (s - x_k)\vec{e}_k)\, \mathrm{d}s \tag{19}$$

where $m_k \leqslant \operatorname{Re}\alpha_k < m_k + 1$. The upper index in the formula denotes the fractional order of the partial derivative while the lower index denotes that it was taken with respect to coordinate $x_k$.

Let $x_1, \ldots, x_m$ be a subset of coordinates in our $n$-dimensional model for which the fractional partial derivatives (19) appear in the equation. Then we define multiplication of functions using the Laplace convolution in the following form.

**Definition 2.3.** *The algebra of functions is defined by the multiplication formula*

$$f * g(\vec{x}) := \int_0^{x_1} \ldots \int_0^{x_m} f\left(\vec{x} - \sum_{l=1}^{m} s_l \vec{e}_l\right) g\left(\vec{x} + \sum_{l=1}^{m} (s_l - x_l)\vec{e}_l\right) \mathrm{d}s_1 \cdots \mathrm{d}s_m \tag{20}$$

*where* $(\vec{e}_l)_k = \delta_{lk}$.

Similarly to the one-dimensional case, the multiplication (20) is associative and commutative.

In the above algebra of functions, Leibniz's rule (18) given by corollary 2.2 is valid for functions fulfilling the respective assumptions concerning their behaviour at $x_k = 0$

$$D_k^{\alpha_k} f * g = \beta_k \left(D_k^{\alpha_k} f\right) * g + (1 - \beta_k) f * D_k^{\alpha_k} g \tag{21}$$

with $\beta_k \in [0, 1]$ for $k = 1, \ldots, m$.

For classical derivatives acting by assumption in directions $j = m + 1, \ldots, n$ we obtain for convolution (20) the standard form of Leibniz's rule

$$\partial_j (f * g) = (\partial_j f) * g + f * \partial_j g. \tag{22}$$

## 3. Stationarity–conservation laws for some fractional partial equations

We discuss the general construction of stationarity–conservation laws assuming that regular—in the sense of lemma 2.1—solutions of the respective fractional differential equations exist at least in a certain area of space.

### *3.1. Mixed fractional differential and differential partial equations*

Let us consider the general equation which contains the fractional and differential parts of the following form:

$$\Lambda(D, \partial)\phi = [\tilde{\Lambda}(D) + \Lambda(\partial)]\phi = \left(\sum_{k=1}^{m} \tilde{\Lambda}_k D_k^{\alpha_k} + \sum_{l=1}^{N} \Lambda_{\mu_1 \cdots \mu_l} \partial^{\mu_1} \cdots \partial^{\mu_l} + \Lambda_0\right)\phi = 0. \tag{23}$$

We study the construction for the homogeneous form of the equation remembering that the addition of the initial terms restricts only the area of application of the stationarity equation and does not change the general construction. We assume that for given variables $x_1, \ldots, x_m$ the equation includes only fractional derivatives in $\tilde{\Lambda}(D)$ while for the remaining coordinates $x_{m+1}, \ldots, x_n$ only partial derivatives appear in the operator $\Lambda(\partial)$.

The coefficients (we also allow matrices) of the equation depend on coordinates and should obey the restriction in the form $(l = 1, \ldots, N)$

$$\partial^{\mu_1} \Lambda_{\mu_1 \cdots \mu_l} = 0 \tag{24}$$

$$\partial^k \Lambda_{\mu_1 \cdots \mu_l} = 0 \qquad k = 1, \ldots, m \tag{25}$$

$$\partial^k \tilde{\Lambda}_j = 0 \qquad k, j = 1, \ldots, m. \tag{26}$$

In addition, as partial derivatives commute, we expect the coefficients $\Lambda$ to be symmetric with respect to the permutation of each set of indices $(\mu_1 \cdots \mu_l)$.

To derive the stationarity–conservation law, we use the Takahashi–Umezawa method [6, 7] (which we have extended to equations with variable coefficients) for the differential part $\Lambda(\partial)$ and the fractional Leibniz's rule (18) for the part $\tilde{\Lambda}(D)$ containing fractional operators.

Each direction of the space yields the component of the current which for coordinates $x_1, \ldots, x_k$ is given by the $\tilde{\Gamma}$ operator of the form

$$\tilde{\Gamma}_k = 2\tilde{\Lambda}_k \tag{27}$$

while for the part $j = m+1, \ldots, n$, we obtain [7]

$$\Gamma_j = \sum_{l=1}^{N-1} \sum_{k=1}^{l} \Lambda_{j\mu_1\cdots\mu_l} (-\overleftarrow{\partial}^{\mu_1}) \cdots (-\overleftarrow{\partial}^{\mu_k}) \partial^{\mu_{k+1}} \cdots \partial^{\mu_l}. \tag{28}$$

It is a well-known fact that, for an arbitrary pair of functions $f$ and $g$, the operator $\Gamma$ fulfils the equality

$$\sum_{j=m+1}^{n} \partial^j f * \Gamma_j g = -f\Lambda(-\overleftarrow{\partial}) * g + f * \Lambda(\partial) g \tag{29}$$

where the multiplication is given by the convolution (20) and $\Lambda(-\overleftarrow{\partial})$ is the conjugated operator for $\Lambda(\partial)$ acting on the left-hand side.

The above property of the $\Gamma$ operator together with Leibniz's rule (18) for fractional derivatives (taken with parameters $\beta_k = \frac{1}{2}, k = 1, \ldots, m$) implies that the following proposition is valid.

**Proposition 3.1.** *Let the function $\phi$ be a solution of equation (23) and let $\phi'$ solve the conjugated equation*

$$\begin{aligned} \phi'\Lambda(-\overleftarrow{D}, -\overleftarrow{\partial}) &= \phi'[\tilde{\Lambda}(-\overleftarrow{D}) + \Lambda(-\overleftarrow{\partial})] \\ &= \phi' \left( -\sum_{k=1}^{m} \tilde{\Lambda}_k \overleftarrow{D}_k^{\alpha_k} + \sum_{l=1}^{N} \Lambda_{\mu_1\ldots\mu_l} (-\overleftarrow{\partial}^{\mu_1}) \cdots (-\overleftarrow{\partial}^{\mu_l}) + \Lambda_0 \right) = 0. \end{aligned} \tag{30}$$

*Then the current given by the components*

$$J_k = \phi' * \tilde{\Gamma}_k \phi \qquad k = 1, \ldots, m \tag{31}$$

$$J_j = \phi' * \Gamma_j \phi \qquad j = m+1, \ldots, n \tag{32}$$

*fulfils the stationarity–conservation equation*

$$\sum_{k=1}^{m} D_k^{\alpha_k} J_k + \sum_{j=m+1}^{n} \partial^j J_j = 0 \tag{33}$$

*provided the solutions $\phi$ and $\phi'$ fulfil the conditions of lemma 2.1 in the neighbourhood of $x_k = 0, k = 1, \ldots, m$.*

**Proof.** We check the law (33) explicitly:

$$\begin{aligned} &\sum_{k=1}^{m} D_k^{\alpha_k} J_k + \sum_{j=m+1}^{n} \partial^j J_j \\ &\quad = \sum_{k=1}^{m} D_k^{\alpha_k} (\phi' * \tilde{\Lambda}_k \phi + \phi' \tilde{\Lambda}_k * \phi) + \sum_{j=m+1}^{n} \partial^j (\phi' * \Gamma_j \phi) \\ &\quad = \sum_{k=1}^{m} \left( D_k^{\alpha_k} \phi' \right) \tilde{\Lambda}_k * \phi + \sum_{k=1}^{m} \phi' * \tilde{\Lambda}_k D_k^{\alpha_k} \phi - \phi'\Lambda(-\overleftarrow{\partial}) * \phi + \phi' * \Lambda(\partial)\phi \\ &\quad = -\phi'\Lambda(-\overleftarrow{D}, -\overleftarrow{\partial}) * \phi + \phi' * \Lambda(D, \partial)\phi = 0. \end{aligned}$$

Thus, for every equation of the form (23) we can produce an exact form of the stationary–conserved current provided the initial equation and its conjugation have solutions which fulfil the asymptotic conditions at $x_k = 0, k = 1, \ldots, m$ allowing the application of Leibniz's rule for fractional partial derivatives $D_k^{\alpha_k}$.

Let us point out that the conjugated equations (30) and (41) are analogues of the characteristic equation for the conservation laws from the theory of partial differential equations [2]. Each solution of the characteristic equation yields a conserved current. In the case of models including fractional derivatives we call this the stationarity–conservation law.

The stationarity–conservation equation (33) can be rewritten in the form of the standard conservation law for modified components of the above current ($m_k < \alpha_k < m_k + 1, k = 1, \ldots, m$)

$$J'_k = (\partial^k)^{m_k}\big(J_k *_k \Phi_{\alpha_k - m_k}\big) \qquad k = 1, \ldots, m \tag{34}$$

$$J'_j = J_j \qquad j = m+1, \ldots, n \tag{35}$$

where the convolution $*_k$ is given by the formula

$$f *_k g(\vec{x}) = \int_0^{x_k} f(\vec{x} - s\vec{e}_k) g(\vec{x} + (s - x_k)\vec{e}_k)\, \mathrm{d}s_k. \tag{36}$$

The new current $J'$ obeys the conservation law

$$\sum_{l=1}^{n} \partial^l J'_l = 0. \tag{37}$$

□

*3.2. Mixed fractional sequential and differential partial equations*

In the previous construction we have considered the fractional part of the operator including only the first power of the corresponding partial fractional derivatives, while in the differential part we have taken an arbitrary polynomial of partial derivatives. We extend the derivation of the stationarity–conservation laws to the general case containing both the polynomial of fractional derivatives and the polynomial of classical partial derivatives

$$\Lambda(D, \partial)\phi = [\tilde{\Lambda}(D) + \Lambda(\partial)]\phi$$
$$= \left(\sum_{k=1}^{M} \tilde{\Lambda}_{\rho_1\cdots\rho_k} D_{\rho_1}^{\alpha_1}\cdots D_{\rho_k}^{\alpha_k} + \sum_{l=1}^{N} \Lambda_{\mu_1\cdots\mu_l}\partial^{\mu_1}\cdots\partial^{\mu_l} + \Lambda_0\right)\phi = 0. \tag{38}$$

The derivatives with respect to the coordinates $x_1, \ldots, x_m$ are the fractional $D_{\rho_i}^{\alpha_i}$ where the upper index denotes the fractional order and the lower index denotes the respective partial direction. The part depending on fractional derivatives now has the form of a partial sequential fractional operator generalizing the sequential operator for one-dimensional space [13]. The coefficients $\Lambda$ and $\tilde{\Lambda}$ are again functions or matrices of functions obeying the main condition (24)–(26). As the derivatives with respect to different coordinates do commute, both types of coefficients are fully symmetric with respect to the permutation of the set of indices.

To obtain the $\Gamma$ operator fulfilling equation (29) we again use the extended Takahashi–Umezawa method for the differential part $\Lambda(\partial)$ and we obtain the components $\Gamma_j$ as given by equation (28) whereas for $\tilde{\Gamma}$ we have

$$\tilde{\Gamma}_k = 2\sum_{j=1}^{M-1}\sum_{l=1}^{j} \tilde{\Lambda}_{k\rho_1\cdots\rho_j}\left(-\overleftarrow{D}_{\rho_1}^{\alpha_1}\right)\cdots\left(-\overleftarrow{D}_{\rho_l}^{\alpha_l}\right) D_{\rho_{l+1}}^{\alpha_{l+1}}\cdots D_{\rho_j}^{\alpha_j}. \tag{39}$$

It is easy to check the analogue of formula (29) for the operator $\tilde{\Gamma}$

$$\sum_{k=1}^{m} D_k^{\alpha_k}(f * \tilde{\Gamma}_k g) = -f\tilde{\Lambda}(-\overleftarrow{D}) * g + f * \tilde{\Lambda}(D)g \tag{40}$$

for an arbitrary pair of functions $f$ and $g$ allowing the application of Leibniz's rule (18) together with their fractional derivatives $D_{\rho_{l+1}}^{\alpha_{l+1}} \cdots D_{\rho_j}^{\alpha_j} g$ and $f\big(-\overleftarrow{D}_{\rho_1}^{\alpha_1}\big) \cdots \big(-\overleftarrow{D}_{\rho_l}^{\alpha_l}\big)$.

The following proposition generalizes the result obtained for the equation with constant coefficients [17].

**Proposition 3.2.** *Let the function $\phi$ be a solution of equation (38) and let $\phi'$ be a solution of the conjugated equation in the form*

$$\begin{aligned} 0 &= \phi'\Lambda(-\overleftarrow{D}, -\overleftarrow{\partial}) \\ &= \phi'\left(\sum_{k=1}^{M} \tilde{\Lambda}_{\mu_1\cdots\mu_k}\left(-\overleftarrow{D}_{\mu_1}^{\alpha_1}\right)\cdots\left(-\overleftarrow{D}_{\mu_k}^{\alpha_k}\right) + \sum_{l=1}^{N} \Lambda_{\mu_1\cdots\mu_l}(-\overleftarrow{\partial}^{\mu_1})\cdots(-\overleftarrow{\partial}^{\mu_l}) + \Lambda_0\right). \end{aligned} \tag{41}$$

*Then the current with the following components,*

$$J_k = \phi' * \tilde{\Gamma}_k\phi \qquad k = 1, \ldots, m \tag{42}$$

$$J_j = \phi' * \Gamma_j\phi \qquad j = m+1, \ldots, n \tag{43}$$

*obeys the stationarity–conservation equation*

$$\sum_{k=1}^{m} D_k^{\alpha_k} J_k + \sum_{j=m+1}^{n} \partial^j J_j = 0 \tag{44}$$

*provided the solutions $\phi$ and $\phi'$, together with their derivatives appearing in the formulae for components (42), fulfil the conditions of lemma 2.1 in the neighbourhood of $x_k = 0, k = 1, \ldots, m$.*

**Proof .** We use the properties of the solutions and of the operators $\Gamma$ and $\tilde{\Gamma}$ and obtain

$$\sum_{j=m+1}^{n} \partial^j J_j = \sum_{j=m+1}^{n} \partial^j(\phi' * \Gamma_j\phi) = -\phi'\Lambda(-\overleftarrow{\partial}) * \phi + \phi' * \Lambda(\partial)\phi$$

$$\sum_{k=1}^{m} D_k^{\alpha_k} J_k = \sum_{k=1}^{m} D_k^{\alpha_k}(\phi' * \tilde{\Gamma}_k\phi) = -\phi'\tilde{\Lambda}(-\overleftarrow{D}) * \phi + \phi' * \tilde{\Lambda}(D)\phi.$$

Thus the left-hand side of the stationarity–conservation formula is of the form

$$\begin{aligned} &\sum_{k=1}^{m} D_k^{\alpha_k} J_k + \sum_{j=m+1}^{n} \partial^j J_j \\ &\quad = -\phi'(\tilde{\Lambda}(-\overleftarrow{D}) + \Lambda(-\overleftarrow{\partial}) + \Lambda_0) * \phi + \phi' * (\tilde{\Lambda}(D) + \Lambda(\partial) + \Lambda_0)\phi = 0 \end{aligned}$$

and vanishes on shell.

We can rewrite the stationarity–conservation law to have the conservation law connected with equation (38). To this aim we apply the definition of the Riemann–Liouville fractional derivative (19). The modified components of the current have a form similar to that derived in the previous section ($m_k < \alpha_k < m_k + 1, k = 1, \ldots, m$)

$$J_k' = (\partial^k)^{m_k} \left( J_k *_k \Phi_{\alpha_k - m_k} \right) \qquad k = 1, \ldots, m \tag{45}$$

$$J_j' = J_j \qquad j = m+1, \ldots, n \tag{46}$$

with the convolution $*_k$ given by equation (36).

They obey the conservation law

$$\sum_{l=1}^{n} \partial^l J_l' = 0. \tag{47}$$

□

*3.3. Stationary and conserved charges for mixed fractional–differential models*

Let us assume that the stationarity–conservation law for a given model is valid in the whole space of coordinates.

Two cases should be considered: when the time derivative in the operator of the equation is a fractional and when it is standard partial one.

Let us assume that the time derivative in equations (23) and (38) is a fractional one. Integrating the time component of the current fulfilling the stationarity–conservation equation (33) and (44) we arrive at the charge

$$Q = \int_{R^{n-1}} \mathrm{d}\vec{x}\; J_t(\vec{x}, t) \tag{48}$$

which is a stationary function of the order $\alpha_t$ which also determines the order of the fractional time derivative

$$D_t^{\alpha_t} Q = 0 \tag{49}$$

provided the respective boundary terms vanish.

For components $J_j$, $j = m+1, \ldots, n$, this means that they vanish at infinity in the respective $j$-direction while the components $J_k$, $k = 2, \ldots, m$, obey the asymptotic condition

$$\lim_{|x_k| \to \infty} (\partial^k)^{m_k} \left( J_k *_k \Phi_{\alpha_k - m_k} \right) = 0 \tag{50}$$

where $m_k < \alpha_k < m_k + 1$.

The above fractional stationarity proposed by Hilfer [49] generalizes constant functions. In fact, the stationary charge fulfilling (49) is a linear combination of power functions with exponents depending on a value of $\alpha_t$.

The second case is the model with a standard time derivative. Then the charge

$$Q = \int_{R^{n-1}} \mathrm{d}\vec{x}\; J_t(\vec{x}, t) \tag{51}$$

is a strictly stationary function of time, which means that it is a true constant function

$$\partial^t Q = 0 \tag{52}$$

when the asymptotic conditions described above for respective components of the currents are fulfilled.

The exact form of the symmetry algebra of the equations (23) and (38) vary for different examples. Let us, however, notice that it includes, for all of these, the momenta

$$P_k = D_k^{\alpha_k} \qquad k = 1, \ldots, m \tag{53}$$

$$P_j = \partial^j \qquad j = m+1, \ldots, n \tag{54}$$

as they commute with the operator of these equations.

However, if we propose to use the above momenta in derivation of conserved currents and charges, we must additionally assume the regular behaviour of the $W(D)P_k\phi$ and $W(D)P_j\phi$ functions in the neighbourhood of zero with respect to the $x_1, \ldots, x_m$ coordinates ($W(D)$ denote the polynomials of fractional derivatives appearing in the formula for $\tilde{\Gamma}$ operator).

When this assumption is fulfilled, the stationary–conserved currents yield the charges

$$Q^\delta = \int_{R^{n-1}} \mathrm{d}\vec{x}\, \phi' * \tilde{\Gamma}_t \delta\phi \qquad (55)$$

for the case where the time derivative is fractional and for the standard time derivative, respectively,

$$Q^\delta = \int_{R^{n-1}} \mathrm{d}\vec{x}\, \phi' * \Gamma_t \delta\phi \qquad (56)$$

where $\delta$ is one of the momentum operators given in equations (53) and (54).

## 4. Applications

### *4.1. Fractional equation for anomalous diffusion*

Let us start the application of the proposed procedure with the fractional equation describing anomalous diffusion. This has been discussed in the general form by Metzler *et al* [26], and earlier in [24, 25], and includes the fractional time derivative

$$\left[D_t^\alpha - \frac{1}{r^{D-1}}\partial^r r^{-\Theta} r^{D-1}\partial^r - \frac{\beta}{r^2}\right] P(r,t) = 0 \qquad (57)$$

where $D$ is the Hausdorff dimension of the underlying fractal structure, $\Theta$ is connected with the anomalous diffusion parameter $d_w = 2 + \Theta$ and $\alpha = 2/d_w$.

The conjugated equation (30) looks as follows for the considered model:

$$\tilde{P}(r,t)\left[-\overleftarrow{D}_t^\alpha - \frac{1}{r^{D-1}}\overleftarrow{\partial}^r r^{-\Theta} r^{D-1}\overleftarrow{\partial}^r - \frac{\beta}{r^2}\right] = 0. \qquad (58)$$

Both equations (57) and (58) should be modified in order to obtain the form obeying the main restriction for variable coefficients (24)–(26). The new density functions are connected with probability density functions $P$ and $\tilde{P}$ by the formula

$$P(r,t) = \rho(r)W(r,t) \qquad \rho(r) = r^{\frac{\Theta-D+1}{2}} \qquad (59)$$

$$\tilde{P}(r,t) = \tilde{\rho}(r)\tilde{W}(r,t) \qquad \tilde{\rho}(r) = r^{\frac{\Theta+D-1}{2}} \qquad (60)$$

$$\rho(r)\tilde{\rho}(r) = r^\Theta. \qquad (61)$$

Written for modified functions $W$ and $\tilde{W}$, equations (57) and (58) look as follows,

$$\left[r^\Theta D_t^\alpha - (\partial^r)^2 - V(r)\right] W(r,t) = 0 \qquad (62)$$

$$\tilde{W}(r,t)\left[-r^\Theta \overleftarrow{D}_t^\alpha - (\overleftarrow{\partial}^r)^2 - V(r)\right] = 0 \qquad (63)$$

with the potential $V$ in the form

$$V(r) = r^{-2}\left[\frac{\Theta - D + 1}{2} + \frac{(\Theta - D + 1)^2}{4}\right] + \beta r^{\Theta-2}. \qquad (64)$$

It is clear that the modified equations (62) and (63) fulfil the main conditions (24)–(26), namely

$$\partial^r \Lambda_{rr} = 0 \qquad \partial^t \Lambda_{rr} = 0 \qquad \partial^t \tilde{\Lambda}_t = 0. \qquad (65)$$

Thus, proposition 3.1 can be applied to obtain the components of the $\Gamma$ operator

$$\tilde{\Gamma}_t = 2r^{\Theta} \tag{66}$$

$$\Gamma_r = \overleftarrow{\partial}^r - \partial^r \tag{67}$$

and the components of the current

$$J_t = \tilde{W}(r,t) * \tilde{\Gamma}_t W(r,t) = 2\tilde{P}(r,t) * P(r,t) \tag{68}$$

$$J_r = \tilde{W}(r,t) * \Gamma_r W(r,t) = \frac{1-D}{r^{\Theta+1}} \tilde{P}(r,t) * P(r,t) + \tilde{P}(r,t) * \Gamma_r P(r,t) \tag{69}$$

where we understand the convolution $*$ in the sense of definition (20), which means that it is taken with respect to the time coordinate.

The above current fulfils the stationarity–conservation law

$$D_t^{\alpha} J_t + \partial^r J_r = 0 \tag{70}$$

and after modification using formulae (34) and (35) ($m < \alpha < m+1$)

$$J_t' = J_t * \frac{t^{-\alpha+m}}{\Gamma(m+1-\alpha)} \tag{71}$$

$$J_r' = J_r \tag{72}$$

we arrive at the conservation law

$$\partial^t J_t' + \partial^r J_r' = 0 \tag{73}$$

which is valid together with the stationarity–conservation law (70) in the area where the solutions $W$ and $\tilde{W}$ used in the construction obey the asymptotic conditions of lemma 2.1.

Let us now assume that the solutions obey the above condition in the whole space. Then the charge

$$Q = \int \mathrm{d}r\, J_t = \int \mathrm{d}r\, 2\tilde{P}(r,t) * P(r,t) \tag{74}$$

is a stationary quantity of the order $\alpha$

$$D_t^{\alpha} Q = 0. \tag{75}$$

### 4.2. 1 + 1 *fractional Fokker–Planck equation*

Let us now study the fractional Fokker–Planck equation. This equation describes anomalous diffusion in an external force field and close to the thermal equilibrium. Widely discussed in the literature [32–36] (see also references therein) it applies to the case when the process has started at $t = 0$, the external field is weak and its influence on the waiting time density is negligible [32]. Written in standard homogeneous form it reads as follows,

$$\left[\partial^t - D_t^{1-\alpha} K_{\alpha} \left((\partial^x)^2 - \partial^x \frac{F(x)}{k_B T}\right)\right] P(x,t) = 0 \tag{76}$$

where $D_t^{1-\alpha}$ is the fractional Riemann–Liouville derivative (13) with $\alpha \in (0,1)$, $(k_B T)^{-1}$ is the Boltzmann factor and $K_{\alpha}$ is the generalized diffusion coefficient.

The above equation is also investigated in the non-homogeneous version containing the initial value of the probability density function

$$\left[D_t^\alpha - K_\alpha\left((\partial^x)^2 - \partial^x\frac{F(x)}{k_B T}\right)\right]P(x,t) = P_0(x)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{77}$$

The conjugated counterpart of the fractional Fokker–Planck equation reads as follows:

$$\tilde{P}(x,t)\left[-\overleftarrow{D}_t^\alpha - K_\alpha\left((\overleftarrow{\partial}^x)^2 + \overleftarrow{\partial}^x\frac{F(x)}{k_B T}\right)\right] = \tilde{P}_0(x)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{78}$$

We should notice here that equations (77) and (78) do not satisfy the restrictions (24)–(26) for coefficients. Thus, both of these should be reformulated similarly to the previous example,

$$P(x,t) = \rho(x)W(x,t) \qquad \rho(x) = \exp\left(\int\frac{F(x)}{2k_B T}\,\mathrm{d}x\right) \tag{79}$$

$$\tilde{P}(x,t) = \tilde{\rho}(x)\tilde{W}(x,t) \qquad \tilde{\rho}(x) = \exp\left(-\int\frac{F(x)}{2k_B T}\,\mathrm{d}x\right) \tag{80}$$

$$\rho(x)\tilde{\rho}(x) = 1. \tag{81}$$

Equations (77) and (78) written for the modified density functions obey the condition (24)–(26) and look as follows:

$$\left[D_t^\alpha - K_\alpha(\partial^x)^2 - V(x)\right]W(x,t) = \frac{P_0(x)}{\rho(x)}\cdot\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{82}$$

$$\tilde{W}(x,t)\left[-\overleftarrow{D}_t^\alpha - K_\alpha(\overleftarrow{\partial}^x)^2 - V(x)\right] = \frac{\tilde{P}_0(x)}{\tilde{\rho}(x)}\cdot\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{83}$$

$$V(x) = -K_\alpha\left[\left(\frac{F(x)}{2k_B T}\right)' + \left(\frac{F(x)}{2k_B T}\right)^2\right]. \tag{84}$$

Following the general method for equations of type (23) we arrive at the components of the $\Gamma$ operator

$$\tilde{\Gamma}_t = 2 \qquad \Gamma_x = K_\alpha\overleftarrow{\partial}^x - K_\alpha\partial^x. \tag{85}$$

In the following we obtain the components of the current:

$$J_t = \tilde{W}(x,t) * \tilde{\Gamma}_t W(x,t) = 2\tilde{P}(x,t) * P(x,t) \tag{86}$$

$$\begin{aligned} J_x &= \tilde{W}(x,t) * \Gamma_x W(x,t) \\ &= \frac{K_\alpha F(x)}{k_B T}\tilde{P}(x,t) * P(x,t) + \tilde{P}(x,t) * \Gamma_x P(x,t). \end{aligned} \tag{87}$$

The derived current obeys the stationarity–conservation law

$$D_t^\alpha J_t + \partial^x J_x = 0 \tag{88}$$

in the area where $\frac{P_0(x)}{\rho(x)} = \frac{\tilde{P}_0(x)}{\tilde{\rho}(x)} = 0$ and solutions $W$, $\tilde{W}$ fulfil the asymptotic conditions from lemma 2.1:

$$\lim_{t\to 0+0} W(x,t) * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \lim_{t\to 0+0}\tilde{W}(x,t) * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = 0. \tag{89}$$

The current (86) and (87) yields the conserved one

$$J_t' = J_t * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \qquad J_x' = J_x \tag{90}$$

which obeys the conservation law

$$\partial^t J_t' + \partial^x J_x' = 0 \tag{91}$$

valid in the area described above for the stationarity–conservation law (88).

*4.3. d + 1 fractional Fokker–Planck equation*

The generalized fractional Fokker–Planck equation [38] in the non-homogeneous version takes the form

$$\left[D_t^\alpha - K_\alpha \triangle + \frac{K_\alpha}{k_B T}\vec{\partial}\cdot\vec{F}(\vec{x})\right] P(\vec{x},t) = P_0(\vec{x})\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{92}$$

with $\alpha \in (0, 1)$, $\triangle = \vec{\partial}\cdot\vec{\partial}$ and the external force field determined by the potential, $\vec{F} = \vec{\partial}U$.

The conjugated counterpart for equation (92) looks as follows:

$$\tilde{P}(\vec{x},t)\left[-\overleftarrow{D}_t^\alpha - K_\alpha \overleftarrow{\triangle} - \frac{K_\alpha}{k_B T}\overleftarrow{\vec{\partial}}\cdot\vec{F}(\vec{x})\right] = \tilde{P}_0(\vec{x})\frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{93}$$

Similarly to the (1 + 1)-dimensional case, we modify the probability density functions

$$P(\vec{x},t) = \rho(\vec{x})W(\vec{x},t) \qquad \rho(\vec{x}) = \exp\left(\frac{U(\vec{x})}{2k_B T}\right) \tag{94}$$

$$\tilde{P}(\vec{x},t) = \tilde{\rho}(\vec{x})\tilde{W}(\vec{x},t) \qquad \tilde{\rho}(\vec{x}) = \exp\left(-\frac{U(\vec{x})}{2k_B T}\right) \tag{95}$$

$$\rho(\vec{x})\tilde{\rho}(\vec{x}) = 1. \tag{96}$$

Equations (92) and (93) reformulated for the new density functions satisfy the main condition (24)

$$\left[D_t^\alpha - K_\alpha\triangle - V(\vec{x})\right] W(\vec{x},t) = \frac{P_0(\vec{x})}{\rho(\vec{x})}\cdot\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{97}$$

$$\tilde{W}(\vec{x},t)\left[-\overleftarrow{D}_t^\alpha - K_\alpha\overleftarrow{\triangle} - V(\vec{x})\right] = \frac{\tilde{P}_0(\vec{x})}{\tilde{\rho}(\vec{x})}\cdot\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{98}$$

$$V(\vec{x}) = -K_\alpha\left[\vec{\partial}\cdot\frac{\vec{F}}{2k_B T} + \frac{\vec{F}}{2k_B T}\cdot\frac{\vec{F}}{2k_B T}\right]. \tag{99}$$

Following proposition 3.1, we obtain the components of the $\Gamma$ operator ($j = 1, \ldots, d$)

$$\tilde{\Gamma}_t = 2 \tag{100}$$

$$\Gamma_j = K_\alpha\overleftarrow{\partial}^j - K_\alpha\partial^j \tag{101}$$

which respectively give the components of the current

$$J_t = \tilde{W}(\vec{x},t) * \tilde{\Gamma}_t W(\vec{x},t) = 2\tilde{P}(\vec{x},t) * P(\vec{x},t) \tag{102}$$

$$\begin{aligned} J_j &= \tilde{W}(\vec{x},t) * \tilde{\Gamma}_j W(\vec{x},t) \\ &= \frac{K_\alpha F_j(\vec{x})}{k_B T}\tilde{P}(\vec{x},t) * P(\vec{x},t) + \tilde{P}(\vec{x},t) * \Gamma_j P(\vec{x},t) \end{aligned} \tag{103}$$

where the convolution is taken with respect to the time coordinate according to the definition (20).

The constructed current obeys the stationarity–conservation law

$$D_t^\alpha J_t + \vec{\partial}\cdot\vec{J} = 0 \tag{104}$$

in the area where the initial conditions vanish

$$\frac{P_0(\vec{x})}{\rho(\vec{x})} = \frac{\tilde{P}_0(\vec{x})}{\tilde{\rho}(\vec{x})} = 0 \tag{105}$$

and the asymptotic conditions of lemma 2.1 are fulfilled,

$$\lim_{t\to 0+0} W(\vec{x},t) * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \lim_{t\to 0+0} \tilde{W}(\vec{x},t) * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = 0. \tag{106}$$

After modification of the time component of the current (102)

$$J'_t = J_t * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \qquad J'_j = J_j \tag{107}$$

we obtain the conservation law

$$\partial^t J'_t + \vec{\partial} \cdot \vec{J}' = 0 \tag{108}$$

valid in the area defined by the conditions (105) and (106).

### *4.4. Fractional Klein–Kramers equation*

The fractional Fokker–Planck equation considered in the previous sections can be deduced from the fractional Klein–Kramers equation [36, 37]. This is a bivariate equation describing the motion of a test particle of mass $m$ under the influence of an external force field $F$ in phase (position-velocity) space. The fractional form of the equation is connected with diverging characteristic time

$$\left[\partial^t - D_t^{1-\alpha}\left(-v^*\partial^x + \partial^v\left(\eta^* v - \frac{F^*(x)}{m}\right) + \frac{\eta^* k_B T}{m}(\partial^v)^2\right)\right] P(x,v,t) = 0 \tag{109}$$

where the following notation is used

$$v^* = v\frac{\tau^*}{\tau^\alpha} \qquad \eta^* = \eta\frac{\tau^*}{\tau^\alpha} \qquad F^*(x) = F(x)\frac{\tau^*}{\tau^\alpha}. \tag{110}$$

Here, $\eta$ denotes the friction constant, $k_B T$ is the Boltzmann temperature, $\tau^*$ is the mean time step, $\tau$ is the internal timescale, $\alpha \in (0, 1)$.

Written in non-homogeneous form, the Klein–Kramers equation looks as follows:

$$\left[D_t^{\alpha} - \left(-v^*\partial^x + \partial^v\left(\eta^* v - \frac{F^*(x)}{m}\right) + \frac{\eta^* k_B T}{m}(\partial^v)^2\right)\right] P(x,v,t) = P_0(x,v) \cdot \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{111}$$

The conjugated partner has the following form according to formula (30):

$$\left[-D_t^{\alpha} - \left(v^*\partial^x - \left(\eta^* v - \frac{F^*(x)}{m}\right)\partial^v + \frac{\eta^* k_B T}{m}(\partial^v)^2\right)\right] \tilde{P}(x,v,t) = \tilde{P}_0(x,v) \cdot \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{112}$$

The above equations do not obey the conditions (24)–(26) describing the admissible variable coefficients. Thus, we modify the probability density functions

$$P(x,v,t) = \rho(v) W(x,v,t) \qquad \rho(v) = \exp\left(\frac{mv^2}{4k_B T}\right) \tag{113}$$

$$\tilde{P}(x,v,t) = \tilde{\rho}(v) \tilde{W}(x,v,t) \qquad \tilde{\rho}(v) = \exp\left(-\frac{mv^2}{4k_B T}\right) \tag{114}$$

$$\rho(v)\tilde{\rho}(v) = 1. \tag{115}$$

Written for the above modified density functions, the fractional Klein–Kramers equation and its conjugation take the form

$$\left[\frac{\tau^{\alpha}}{\tau^{*}} D_t^{\alpha} + v\partial^{x} + \frac{F(x)}{m}\partial^{v} + \frac{\eta k_B T}{m}(\partial^{v})^2 + V(x, v)\right] W(x, v, t) = \frac{P_0(x, v)}{\rho(v)} \cdot \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \cdot \frac{\tau^{\alpha}}{\tau^{*}} \tag{116}$$

$$\left[-\frac{\tau^{\alpha}}{\tau^{*}} D_t^{\alpha} - v\partial^{x} - \frac{F(x)}{m}\partial^{v} + \frac{\eta k_B T}{m}(\partial^{v})^2 + V(x, v)\right] \tilde{W}(x, v, t) = \frac{\tilde{P}_0(x, v)}{\tilde{\rho}(v)} \cdot \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \cdot \frac{\tau^{\alpha}}{\tau^{*}} \tag{117}$$

with the potential

$$V(x, v) = -\frac{\eta}{2} + \frac{F(x)v}{2k_B T} - \frac{m\eta v^2}{4k_B T}. \tag{118}$$

Analysing the form of the modified Klein–Kramers equation we conclude that it obeys the conditions (24)–(26), namely

$$\partial^{x}\Lambda_x + \partial^{v}\Lambda_v = 0 \qquad \partial^{v}\Lambda_{vv} = 0 \tag{119}$$

$$\partial^{t}\Lambda_v = 0 \qquad \partial^{t}\Lambda_{vv} = 0 \tag{120}$$

$$\partial^{t}\Lambda_x = 0 \qquad \partial^{t}\tilde{\Lambda}_t = 0. \tag{121}$$

Thus, we now apply proposition 3.1 to construct the components of the $\Gamma$ operator

$$\tilde{\Gamma}_t = 2\frac{\tau^{\alpha}}{\tau^{*}} \qquad \Gamma_x = v \qquad \Gamma_v = \frac{F(x)}{m} + \frac{\eta k_B T}{m}(-\overleftarrow{\partial}^{v} + \partial^{v}) \tag{122}$$

which in turn produce the components of the current

$$J_t = \tilde{W}(x, v, t) * \tilde{\Gamma}_t W(x, v, t) = 2\frac{\tau^{\alpha}}{\tau^{*}}\tilde{P}(x, v, t) * P(x, v, t) \tag{123}$$

$$J_x = \tilde{W}(x, v, t) * \Gamma_x W(x, v, t) = v\tilde{P}(x, v, t) * P(x, v, t) \tag{124}$$

$$J_v = \tilde{W}(x, v, t) * \Gamma_v W(x, v, t) = \left(\frac{F(x)}{m} - \eta v\right)\tilde{P}(x, v, t) * P(x, v, t) + \frac{\eta k_B T}{m}\tilde{P}(x, v, t) * (-\overleftarrow{\partial}^{v} + \partial^{v})P(x, v, t). \tag{125}$$

The constructed current obeys the stationarity–conservation law

$$D_t^{\alpha} J_t + \partial^{x} J_x + \partial^{v} J_v = 0 \tag{126}$$

in the area of phase space where the initial terms vanish

$$P_0(x, v) = \tilde{P}_0(x, v) = 0 \tag{127}$$

and the asymptotic conditions of lemma 2.1 are fulfilled,

$$\lim_{t\to 0+0} W(x, v, t) * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \lim_{t\to 0+0} \tilde{W}(x, v, t) * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = 0. \tag{128}$$

In the same area given by equations (127) and (128) the modified current

$$J'_t = J_t * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \qquad J'_x = J_x \qquad J'_v = J_v \tag{129}$$

obeys the conservation law

$$\partial^t J'_t + \partial^x J'_x + \partial^v J'_v = 0. \tag{130}$$

### 4.5. Generalized Cattaneo equation

Let us close the review of applications with the generalized Cattaneo equation [27, 28] in (1 + 1)-dimensional space. This equation has constant coefficients, nevertheless it yields an interesting example of mixed fractional sequential and differential model of type (38)

$$\left[D_t^\alpha + \tau^\alpha \left(D_t^\alpha\right)^2 - D(\partial^x)^2\right] P(x,t) = 0 \tag{131}$$

where $\alpha \in (0, 1)$, $D$ is the diffusion constant and $\tau$ is defined by the finite propagation velocity, namely $v = \sqrt{D/\tau}$.

The conjugated Cattaneo equation looks as follows:

$$\tilde{P}(x,t)\left[-\overleftarrow{D}_t^\alpha + \tau^\alpha \left(\overleftarrow{D}_t^\alpha\right)^2 - D(\overleftarrow{\partial}^x)^2\right] = 0. \tag{132}$$

For the Cattaneo equation of type (131) we obtain the components of the $\Gamma$ operator

$$\tilde{\Gamma}_t = -2\tau^\alpha \overleftarrow{D}_t^\alpha + 2\tau^\alpha D_t^\alpha - 2 \tag{133}$$

$$\Gamma_x = D\overleftarrow{\partial}^x - D\partial^x \tag{134}$$

which determine the components of the current

$$J_t = \tilde{P}(x,t) * \tilde{\Gamma}_t P(x,t) \tag{135}$$

$$J_x = \tilde{P}(x,t) * \Gamma_x P(x,t) \tag{136}$$

where the convolution is taken with respect to the time coordinate.

The derived current satisfies the stationarity–conservation law

$$D_t^\alpha J_t + \partial^x J_x = 0. \tag{137}$$

After modification of the time component of the current (135) and (136)

$$J'_t = J_t * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \qquad J'_x = J_x \tag{138}$$

we arrive at the conservation law

$$\partial^t J'_t + \partial^x J'_x = 0. \tag{139}$$

Let us point out that both conservation laws (137) and (139) are valid in the area where the asymptotic conditions

$$\lim_{t\to 0+0} P(x,t) * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \lim_{t\to 0+0} \tilde{P}(x,t) * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = 0 \tag{140}$$

$$\lim_{t\to 0+0} D_t^\alpha P(x,t) * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \lim_{t\to 0+0} D_t^\alpha \tilde{P}(x,t) * \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = 0 \tag{141}$$

are fulfilled according to lemma 2.1.

As the Cattaneo equation has an homogeneous form and the validity domain of the stationarity–conservation law depends only on properties of solutions, we can construct the stationary charge

$$Q = \int \mathrm{d}x\, J_t \qquad D_t^{\alpha} Q = 0 \tag{142}$$

assuming that the solutions obey the conditions (140) and (141) for arbitrary $x \in R$.

## 5. Final remarks

We have developed a method for deriving the stationarity–conservation law for linear fractional differential equations with variable coefficients. The validity domain of these laws is restricted both by the asymptotic properties of solutions and for some models by initial and boundary conditions.

In general, the obtained currents are non-local quantities due to the multiplication of functions defined via Laplace convolution. In this paper, we have focused on models with Riemann–Liouville-type fractional derivatives. Examples containing other fractional derivatives are still under investigation.

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